Absence of finite size correction at the combinatorial point of the integrable higher spin XXZ chain

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Abstract

We investigate the integrable higher spin XXZ chain at the Razumov-Stroganov point. We present a method to evaluate the exact value of the eigenvalue which is conjectured to correspond to the groundstate of the Hamiltonian for finite size chain from the Baxter Q operator. This allows us to examine the exact total energy difference between different number of total sites, from which we find strong evidence for the absence of finite size correction to the groundstate energy.

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1 Introduction

The spin 1/2 Heisenberg XXZ chain under the periodic boundary condition

$$H(\eta) = -\frac{1}{2}\sum_{j=1}^{M}[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \mathrm{ch}\eta(\sigma_j^z\sigma_{j+1}^z - 1)]. \tag{1}$$

is the first discovered and is one of the most fundamental models in quantum integrable systems. The Hamiltonian can be diagonalized by the Bethe ansatz [1] to give the eigenvalues

$$\mathcal{E}(\eta) = \sum_{j=1}^{M}(2\mathrm{ch}\eta - w_j - w_j^{-1}), \tag{2}$$

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under the following constraints between the parameters $\{w_j\}$

$$w_j^M = \prod_{\substack{k=1 \\ k\neq j}}^{p} \left\{ -\frac{w_j w_k - 2\mathrm{ch}(\eta) w_j + 1}{w_j w_k - 2\mathrm{ch}(\eta) w_k + 1} \right\}, \tag{3}$$

which is called the Bethe ansatz equation. The groundstate energy can be calculated exactly in the thermodynamic limit by making Fourier transform, but it is in general impossible to find the exact value for finite size chain, due to finite size correction. However, taking the anisotropy parameter to a special value $\eta = -2\pi i/3$, the total number of sites to be odd $M = 2N+1$ and the total spin to be $S_z^{\mathrm{tot}} = 1/2$ $(p = N)$, the groundstate energy is shown to have no finite size correction [2–4]

$$\mathcal{E}(-2\pi i/3) = -M. \tag{4}$$

For the last ten years, many exact results such as the components of the groundstate wavefunction and the emptiness formation probability were conjectured to be related to combinatorial objects such as the alternating sign matrix [3–6], some of them have been proved [7–9]. Nowadays, the special point of the anisotropy parameter $\eta = -2\pi i/3$ is called the combinatorial point or the Razumov-Stroganov point.

In this paper, we describe a method to evaluate the eigenvalue which is conjectured to correspond to the groundstate of the Hamiltonian of the integrable higher half-integer spin-s XXZ chain of finite size at the Razumov-Stroganov point. By appropriately normalizing the overall factor of the Hamiltonian which does not depend on the length of the chain, the eigenvalue of the Hamiltonian can be calculated as

$$\mathcal{E}(\eta) = \sum_{j=1}^{M} (2\mathrm{ch}(2s\eta) - w_j - w_j^{-1}), \tag{5}$$

under the Bethe ansatz equation

$$w_j^M = \prod_{\substack{k=1 \\ k\neq j}}^{p} \left\{ -\frac{\mathrm{sh}(\eta) w_j w_k - \mathrm{sh}((2s+1)\eta) w_j + \mathrm{sh}((2s-1)\eta) w_k + \mathrm{sh}\eta}{\mathrm{sh}(\eta) w_j w_k - \mathrm{sh}((2s+1)\eta) w_k + \mathrm{sh}((2s-1)\eta) w_j + \mathrm{sh}\eta} \right\}. \tag{6}$$

As mentioned before, one cannnot evaluate the eigenvalue exactly for finite size chain in general. However, for odd number of sites and at the Razumov-Stroganov point, we recently calculated a polynomial called the Q operator whose zeros give the Bethe roots [10]. This was obtained by solving the Baxter TQ equation [11, 12] which is essentially equivalent to the Bethe ansatz equation. The computed Q operator is based on the variables $z_j = (w_j - \mathrm{e}^{-2s\eta})/(\mathrm{e}^{-2s\eta} w_j - 1)$ rather than the variables w_j. To evaluate the exact eigenvalue of the Hamiltonian from the obtained Q operator, we switch from the Q operator in the variables z_j to the one in the variables w_j. This leads to the transformation of the symmetric polynomials of the variables z_j to those of the variables w_j, and the simplest symmetric polynomial $\sum_{j=1}^{p} w_j$ essentially leads to the eigenvalue of the Hamiltonian. By this procedure, we can easily evaluate the groundstate energy exactly for finite chain, and one finds strong evidence for the absence of finite size correction for higher half-integer spin XXZ chain at the Razumov-Stroganov point, generalizing the behavior observed in the spin 1/2 XXZ chain. Calculating

the groundstate eigenvalue naively by solving the zeros of the Q operator just only gives a numerical value. The transformation of the Q operators is essential to extract the exact value, from which one can check the validity of the vanishing of finite size correction. Based on the conjecture for the absence of finite size correction, it is enough to consider only $M = 3$ sites ($N = 1$) and $M = 5$ sites ($N = 2$) to extract the groundstate energy which can be easily evaluated by the Q operator.

2 TQ equation and Q operator

In this section, we give a brief review on the results for the Q operator of higher spin XXZ chain at the Razumov-Stroganov point [10]. In terms of the variables $z_j = (w_j - \mathrm{e}^{-2s\eta})/(\mathrm{e}^{-2s\eta}w_j - 1)$, the Bethe ansatz equation (6) can be expressed as

$$\left(\frac{z_j \mathrm{e}^{2s\eta} - 1}{z_j - \mathrm{e}^{2s\eta}}\right)^M = \prod_{\substack{k=1 \\ k\neq j}}^{p} \left(\frac{z_j \mathrm{e}^{2\eta} - z_k}{z_j - z_k \mathrm{e}^{2\eta}}\right), \tag{7}$$

or, in the familiar form

$$\left(\frac{\mathrm{sh}(u_j + s\eta)}{\mathrm{sh}(u_j - s\eta)}\right)^M = \prod_{\substack{k=1 \\ k\neq j}}^{p} \frac{\mathrm{sh}(u_j - u_k + \eta)}{\mathrm{sh}(u_j - u_k - \eta)}, \tag{8}$$

in terms of the variables u_j related to z_j by $z_j = \exp(2u_j)$.
The Bethe ansatz equation can be obtained from the Baxter's TQ equation

$$T(u)Q(u) = \mathrm{sh}^M(u + s\eta)Q(u - \eta) + \mathrm{sh}^M(u - s\eta)Q(u + \eta) = 0, \tag{9}$$

where $T(u)$ is the transfer matrix whose auxiliary space has spin 1/2 and the quantum space is the M-fold tensor product of spin s spaces. u is the spectral parameter and η is the anisotropy parameter associated with the XXZ chain. The Q operator $Q(u) = \prod_{j=1}^{p} \mathrm{sh}(u - u_j)$ encodes the information of the eigenstate of the model since the TQ equation (9) reduces to the Bethe ansatz equation of the higher spin XXZ chain by setting the spectral parameter u to $u = u_j$. Solving the Bethe ansatz equation is equivalent to computing the Q operator.

For the half-integer spin $s = (L - 2)/2$ ($L = 3, 5, 7, \cdots$) XXZ chain with odd number of total sites $M = 2N + 1$ ($N = 1, 2, 3, \cdots$), the transfer matrix eigenvalue of $T(u)$ which is conjectured to correspond to the groundstate was found to have a simple form at the Razumov-Stroganov point $\eta = -(L-1)\pi i/L$. In the sector with $p = N + (2N+1)(L-3)/2 + m$ Bethe roots, an exact transfer matrix eigenvalue has the following simple form [12]

$$T(u) = 2\mathrm{ch}\left(\frac{(L-1)(1-2m)\pi i}{2L}\right)\mathrm{sh}^M u. \tag{10}$$

To analyze the corresponding Q operator, it is useful to change to the spectral variables

$z = \exp(2u)$ and $z_j = \exp(2u_j)$ and redefine the Q operator as

$$Q(z) = \prod_{j=1}^{p}(z - z_j) = \sum_{j=0}^{p}(-1)^j z^{p-j} e_j^{(p)}, \tag{11}$$

$$e_j^{(p)} = \sum_{1\le i_1<i_2<\cdots<i_j\le p} z_{i_1} z_{i_2}\cdots z_{i_j}\ (j = 1, 2, \cdots, p),\ e_0^{(p)} = 1. \tag{12}$$

We recently calculated the Q operator in two ways. Let us briefly present the results below. We restrict to the case $m = 0$, i.e., in the sector $S_z^{\rm tot} = 1/2$ $(p = N + (2N+1)(L-3)/2)$. First, we showed that solving the TQ equation reduces to solving the following set of linear equations of the elementary symmetric polynomials $e_k^{(p)}$ of z_j

$$\sum_{j=\max(0,\ell-2N-1)}^{\min(N+(2N+1)(L-3)/2,\ell)} \binom{2N+1}{\ell-j} e_j^{(N+(2N+1)(L-3)/2)} = 0, \tag{13}$$

for $\ell = 0, 1, \cdots, NL + (L-1)/2$ $(\ell \neq Lk, Lk + (L-1)/2\ (k = 0, 1, \cdots, N))$.
We also evaluated the Q operator in another way by use of the interpolation formula to find

$$\begin{aligned} Q(z) =&(z-1)^{-(2N+1)}\Bigg\{\sum_{k=0}^{N/2}(-1)^k\binom{N}{k}\prod_{j=0}^{N}\frac{(L-1)/2+Lj}{(L-1)/2-Lk+Lj}(z^{LN+(L-1)/2-Lk}-z^{Lk}) \\ &+\sum_{k=0}^{N/2-1}(-1)^k\binom{N}{k}\prod_{j=0}^{N}\frac{(L-1)/2+Lj}{-(L-1)/2-Lk+Lj}(z^{LN-Lk}-z^{Lk+(L-1)/2})\Bigg\}, \end{aligned} \tag{14}$$

for N even and

$$\begin{aligned} Q(z) =&(z-1)^{-(2N+1)}\Bigg\{\sum_{k=0}^{(N-1)/2}(-1)^k\binom{N}{k}\prod_{j=0}^{N}\frac{(L-1)/2+Lj}{(L-1)/2-Lk+Lj}(z^{LN+(L-1)/2-Lk}-z^{Lk}) \\ &+\sum_{k=0}^{(N-1)/2}(-1)^k\binom{N}{k}\prod_{j=0}^{N}\frac{(L-1)/2+Lj}{-(L-1)/2-Lk+Lj}(z^{LN-Lk}-z^{Lk+(L-1)/2})\Bigg\}, \end{aligned} \tag{15}$$

for N odd.

3 Transformation of Q operators

It was useful to change to the variables z and z_j to calculate the Q operator. On the other hand, it is essential to use the variables $w = (z\mathrm{e}^{2s\eta}-1)/(z-\mathrm{e}^{2s\eta})$ and $w_j = (z_j\mathrm{e}^{2s\eta}-1)/(z_j-\mathrm{e}^{2s\eta})$ to evaluate the exact eigenvalue of the Hamiltonian [4]. This is because $E_1^{(p)} = \sum_{j=1}^{p} w_j$ which forms a part of the eigenvalue (5) is the coefficient of w^{p-1} of the following Q operator in the w variables

$$\chi(w) = \prod_{j=1}^{p}(w - w_j) = \sum_{j=0}^{p}(-1)^j w^{p-j} E_j^{(p)}, \tag{16}$$

$$E_j^{(p)} = \sum_{1\le i_1<i_2<\cdots<i_j\le p} w_{i_1} w_{i_2}\cdots w_{i_j}\ (j = 1, 2, \cdots, p),\ E_0^{(p)} = 1. \tag{17}$$

To evaluate the exact eigenvalue, we need to express $E_1^{(p)}$ in terms of the symmetric polynomials $e_k^{(p)}$ of the variables z_j. We insert the relation

$$w - w_j = \frac{(\mathrm{e}^{-4\pi i/L} - 1)(z - z_j)}{(z - \mathrm{e}^{-2\pi i/L})(\mathrm{e}^{-2\pi i/L} - z_j)}, \tag{18}$$

into (16) and make the following transformation to express the coefficients of the powers of w in terms of $e_k^{(p)}$

$$\begin{aligned}
\chi(w) &= \left(\frac{\mathrm{e}^{-4\pi i/L} - 1}{z - \mathrm{e}^{-2\pi i/L}}\right)^p \frac{Q(z)}{Q(\mathrm{e}^{-2\pi i/L})} \\
&= (w - \mathrm{e}^{-2\pi i/L})^p \frac{Q\left(\frac{\mathrm{e}^{-2\pi i/L}w - 1}{w - \mathrm{e}^{-2\pi i/L}}\right)}{Q(\mathrm{e}^{-2\pi i/L})} \\
&= \frac{(w - \mathrm{e}^{-2\pi i/L})^p}{Q(\mathrm{e}^{-2\pi i/L})} \prod_{j=1}^{p} \left(\frac{\mathrm{e}^{-2\pi i/L}w - 1}{w - \mathrm{e}^{-2\pi i/L}} - z_j\right) \\
&= \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \prod_{j=1}^{p} (\mathrm{e}^{-2\pi i/L}w - 1 - z_j(w - \mathrm{e}^{-2\pi i/L})) \\
&= \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \sum_{k=0}^{p} (\mathrm{e}^{-2\pi i/L}w - 1)^{p-k}(\mathrm{e}^{-2\pi i/L} - w)^k e_k^{(p)} \\
&= \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \sum_{k=0}^{p} \sum_{\ell=0}^{p-k} \sum_{j=0}^{k} (-1)^{p+j-k-\ell} \mathrm{e}^{-2\pi i(k+\ell-j)/L} \binom{p-k}{\ell} \binom{k}{j} e_k^{(p)} w^{j+\ell} \\
&= \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \sum_{k=0}^{p} \sum_{\alpha=0}^{p} \sum_{j=\max(0,k-\alpha)}^{\min(k,p-\alpha)} (-1)^{\alpha+k} \mathrm{e}^{-2\pi i(k+p-\alpha-2j)/L} \binom{p-k}{p-\alpha-j} \binom{k}{j} w^{p-\alpha} e_k^{(p)}.
\end{aligned} \tag{19}$$

Equating the coefficients of the powers of w of (16) and (19), one gets

$$E_\alpha^{(p)} = \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \sum_{k=0}^{p} \sum_{j=\max(0,k-\alpha)}^{\min(k,p-\alpha)} (-1)^k \mathrm{e}^{-2\pi i(k+p-\alpha-2j)/L} \binom{p-k}{p-\alpha-j} \binom{k}{j} w^{p-\alpha} e_k^{(p)}. \tag{20}$$

In particular, we have

$$E_1^{(p)} = \frac{1}{Q(\mathrm{e}^{-2\pi i/L})} \sum_{k=0}^{p} \sum_{j=\max(0,k-1)}^{\min(k,p-1)} (-1)^k \mathrm{e}^{-2\pi i(k+p-1-2j)/L} \binom{p-k}{p-1-j} \binom{k}{j} w^{p-1} e_k^{(p)}. \tag{21}$$

We can furthermore simplify the numerator and the denominator utilizing the relation

$$e_k^{(p)} = (-1)^p e_{p-k}^{(p)}. \tag{22}$$

This relation can be shown by combining

$$e_k^{(p)}|_{z_j \to z_j^{-1}} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le p} z_{i_1}^{-1} z_{i_2}^{-1} \cdots z_{i_k}^{-1} \\ = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le p} z_{i_1} z_{i_2} \cdots z_{i_k} = e_k^{(p)}, \tag{23}$$

which follows from the $z \leftrightarrow z^{-1}, z_j \leftrightarrow z_j^{-1}$ invariance of the TQ equation

$$-2\mathrm{ch}\left(\frac{(L-1)\pi i}{2L}\right)(z-1)^{2N+1} \prod_{j=1}^{N+(2N+1)(L-3)/2} (z - z_j) \\ + \mathrm{e}^{(1-L)\pi i/2L}(z - \mathrm{e}^{2\pi i/L})^{2N+1} \prod_{j=1}^{N+(2N+1)(L-3)/2} (z - \mathrm{e}^{2\pi i/L} z_j) \\ + \mathrm{e}^{(L-1)\pi i/2L}(z - \mathrm{e}^{-2\pi i/L})^{2N+1} \prod_{j=1}^{N+(2N+1)(L-3)/2} (z - \mathrm{e}^{-2\pi i/L} z_j) = 0, \tag{24}$$

and

$$Q(0) = (-1)^p \prod_{j=1}^{p} z_j = 1, \tag{25}$$

which follows by substituting $z = 0$ in (14) and (15), as

$$e_k^{(p)} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le p} z_{i_1} z_{i_2} \cdots z_{i_k} \\ = \prod_{j=1}^{p} z_j \sum_{1 \le i_1 < i_2 < \cdots < i_{p-k} \le p} z_{i_1}^{-1} z_{i_2}^{-1} \cdots z_{i_{p-k}}^{-1} \\ = (-1)^p \sum_{1 \le i_1 < i_2 < \cdots < i_{p-k} \le p} z_{i_1} z_{i_2} \cdots z_{i_{p-k}} \\ = (-1)^p e_{p-k}^{(p)}. \tag{26}$$

Using this relation, the numerator can be expressed as

$$\sum_{k=0}^{p} \sum_{j=\max(0,k-1)}^{\min(k,p-1)} (-1)^k \mathrm{e}^{-2\pi i(k+p-1-2j)/L} \binom{p-k}{p-1-j} \binom{k}{j} w^{p-1} e_k^{(p)} \\ = 2\mathrm{e}^{-\pi p i/L} \sum_{k=0}^{p-1} (-1)^k (p-k) \cos\left(\frac{\pi(2k+2-p)}{L}\right) e_k^{(p)}, \tag{27}$$

and the denominator as

$$Q(\mathrm{e}^{-2\pi i/L}) = \mathrm{e}^{-\pi p i/L} \sum_{k=0}^{p} (-1)^k \cos\left(\frac{\pi(p-2k)}{L}\right) e_k^{(p)}. \tag{28}$$

The denominator can also be evaluated by setting $z = \mathrm{e}^{-2\pi i/L}$ in (14) or (15). The factor $\mathrm{e}^{-\pi p i/L}$ in (27) and (28) cancels out and we have

$$E_1^{(p)} = \sum_{j=1}^{p} w_j = \frac{2\sum_{k=0}^{p-1}(-1)^k(p-k)\cos\left(\frac{\pi(2k+2-p)}{L}\right)e_k^{(p)}}{\sum_{k=0}^{p}(-1)^k\cos\left(\frac{\pi(p-2k)}{L}\right)e_k^{(p)}}. \tag{29}$$

This is the expression of the term $E_1^{(p)} = \sum_{j=1}^{p} w_j$ in terms of the symmetric polynomials $e_k^{(p)}$ of the variables z_k. The other term $\sum_{j=1}^{p} w_j^{-1}$ which consists another part of the eigenvalue of the Hamiltonian (5) gives the same value with $E_1^{(p)}$

$$\sum_{j=1}^{p} w_j = \sum_{j=1}^{p} w_j^{-1}. \tag{30}$$

This can be shown by utilizing the invariance (23) and noting that (29) replaced by $w_j \to w_j^{-1}$ and $z_j \to z_j^{-1}$ holds by comparing $w_j = (z_j\mathrm{e}^{2s\eta}-1)/(z_j-\mathrm{e}^{2s\eta})$ and $w_j^{-1} = (z_j^{-1}\mathrm{e}^{2s\eta}-1)/(z_j^{-1}-\mathrm{e}^{2s\eta})$

$$\begin{aligned}\sum_{j=1}^{p} w_j^{-1} &= \frac{2\sum_{k=0}^{p-1}(-1)^k(p-k)\cos\left(\frac{\pi(2k+2-p)}{L}\right)e_k^{(p)}|_{z_j\to z_j^{-1}}}{\sum_{k=0}^{p}(-1)^k\cos\left(\frac{\pi(p-2k)}{L}\right)e_k^{(p)}|_{z_j\to z_j^{-1}}} \\ &= \frac{2\sum_{k=0}^{p-1}(-1)^k(p-k)\cos\left(\frac{\pi(2k+2-p)}{L}\right)e_k^{(p)}}{\sum_{k=0}^{p}(-1)^k\cos\left(\frac{\pi(p-2k)}{L}\right)e_k^{(p)}} \\ &= \sum_{j=1}^{p} w_j. \end{aligned} \tag{31}$$

4 Absence of finite size correction

The procedure to calculate the exact eigenvalue of the Hamiltonian for finite size chain can be summarized as:

(i)Solve the TQ equation to calculate the Q operator in the z variables. This reduces to solving the linear equations (13) of the symmetric polynomials $e_j^{(p)}$ of the z variables, or to solve the TQ equation explicitly by use of the interpolation formula to get (14) or (15).

(ii)Relate the Q operator in the z variables and the w variables and express the symmetric polynomials $E_j^{(p)}$ of the w variables in terms of the symmetric polynomials $e_k^{(p)}$ of the z variables.

(iii)Evaluate the exact eigenvalue (5) by inserting the exact value of $e_k^{(p)}$ computed in (i) to the expression (29) relating $E_1^{(p)}$ and $e_k^{(p)}$ which is done in (ii).

For spin 1/2 ($L = 3$), it is shown [4] that

$$\sum_{j=1}^{p} w_j = \frac{1}{2} + \frac{N}{2}, \tag{32}$$

by identifying the symmetric polynomials in the w variables with the refined enumuration of the alternating sign matrix. (32) can also be observed by the procedure described above. This leads to the absence of finite size correction to the groundstate energy at the Razumov-Stroganov point

$$\mathcal{E}(-2\pi i/3) = -M. \tag{33}$$

Next, we examine the spin 3/2 ($L = 5$) chain. By calculating $E_1^{(p)}$ for various number of total sites, we conjecture that the following relation

$$\sum_{j=1}^{p} w_j = \frac{1+\sqrt{5}}{2} + \frac{3+5\sqrt{5}}{4}N, \tag{34}$$

holds. This leads to the following conjecture for the eigenvalue

$$\mathcal{E}(-4\pi i/5) = -\frac{3+\sqrt{5}}{2}M, \tag{35}$$

which means there is no finite size correction to the groundstate energy for the spin 3/2 chain at the Razumov-Stroganov point $\eta = -4\pi i/5$ as well.

By examining larger spins, we conjecture that $E_1^{(p)}$ can be expressed as

$$\sum_{j=1}^{p} w_j = A + \left\{ 2A + \cos\left(\frac{2\pi}{L}\right) \right\} N, \tag{36}$$

where A depends only on the spin value. One can see $A = 1/2$ for spin 1/2 and $A = (1+\sqrt{5})/2$ for spin 3/2 as above. We have checked this for various spins and total number of sites. This leads to the absence of finite size correction to the groundstate energy

$$\mathcal{E}(-(L-1)\pi i/L) = \left((L-3)\cos\left(\frac{2\pi}{L}\right) - 2A \right) M. \tag{37}$$

Based on the conjecture for the absence of finite size correction, $E_1^{(p)} = \sum_{j=1}^{p} w_j$ and the groundstate energy can be evaluated by only treating $M = 3$ sites ($N = 1$) and $M = 5$ sites ($N = 2$). Here is a list of $\sum_{j=1}^{p} w_j$ for spin 5/2 ($L = 7$), spin 7/2 ($L = 9$) and spin 9/2 ($L = 11$). Note that the value A in (37) is obtained by substituting $N = 0$ in the expression for $\sum_{j=1}^{p} w_j$ listed below.

spin 5/2 ($L = 7$)

$$\begin{aligned}\sum_{j=1}^{p} w_j =& (N-1)\frac{6(-499 + 525\cos(\pi/7) + 694\sin(\pi/14) - 900\sin(3\pi/14))}{-235 + 290\cos(\pi/7) + 350\sin(\pi/14) - 434\sin(3\pi/14)} \\ &+ (2-N)\frac{-6 + 120\cos(\pi/7) + 81\sin(\pi/14) - 38\sin(3\pi/14)}{-2 + 15\cos(\pi/7) + 12\sin(\pi/14) - 6\sin(3\pi/14)}.\end{aligned} \tag{38}$$

spin 7/2 ($L = 9$)

$$\sum_{j=1}^{p} w_j = (N-1)\frac{7695 + 43820\cos(\pi/9) - 26108\cos(2\pi/9) - 32210\sin(\pi/18)}{351 + 2450\cos(\pi/9) - 1640\cos(2\pi/9) - 1910\sin(\pi/18)} + (2-N)\frac{-459 + 250\cos(\pi/9) - 1360\cos(2\pi/9) - 976\sin(\pi/18)}{-36 + 40\cos(\pi/9) - 124\cos(2\pi/9) - 100\sin(\pi/18)}. \quad (39)$$

spin 9/2 ($L = 11$)

$$\begin{aligned}
&\sum_{j=1}^{p} w_j = (N-1)\frac{B}{C} + (2-N) \\
&\times \frac{-75 + 860\cos(\pi/11) - 207\cos(2\pi/11) + 575\sin(\pi/22) - 378\sin(3\pi/22) + 765\sin(5\pi/22)}{-9 + 60\cos(\pi/11) - 21\cos(2\pi/11) + 45\sin(\pi/22) - 30\sin(3\pi/22) + 55\sin(5\pi/22)}, \\
&B = -8675 + 9780\cos(\pi/11) - 16727\cos(2\pi/11) + 12895\sin(\pi/22) \\
&\quad - 15050\sin(3\pi/22) + 10925\sin(5\pi/22) \\
&C = -376 + 480\cos(\pi/11) - 730\cos(2\pi/11) + 590\sin(\pi/22) \\
&\quad - 670\sin(3\pi/22) + 520\sin(5\pi/22).
\end{aligned} \quad (40)$$

5 Conclusion

In this paper, we described a method to evaluate the groundstate eigenvalue of the integrable higher spin XXZ chain at the Razumov-Stroganov point by the Q operator. Since the spectral variables of the Bethe roots convenient to evaluate the Q operator and the Hamiltonian are different, we made transformation of the Q operators of the two types of spectral variables. This enabled us to get the exact groundstate eigenvalue, which gives strong evidence for the absence of finite size correction for higher half-integer integrable spin chain, generalizing the behavior observed in the spin 1/2 chain. Calculating the groundstate eigenvalue naively by solving the zeros of the Q operator just only gives a numerical value, and the transformation of the Q operators is essential to extract the exact value. For the spin 1/2 chain, this lead to the identification of the symmetric polynomials of the Bethe roots to the refined enumuration of alternating sign matrix [4]. It may be possible to prove for higher spins by considering alternating sign matrix for higher spins. Alternating sign matrix for spin 1 has been considered in [13]. Another promising approach is the investigation from the supersymmetric point of view [14, 15]. The eigenstate absent from finite size correction should correspond to the zero energy groundstate of a Hamiltonian which have supersymmetry. For higher spins, progresses have been made in [14, 16].

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